

Twisted cohomology of configuration spaces and spaces of maximal tori via point-counting

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March 15, 2016

Abstract

We consider two families of algebraic varieties Y_n indexed by natural numbers n : the configuration space of unordered n -tuples of distinct points on \mathbb{C} , and the space of unordered n -tuples of linearly independent lines in \mathbb{C}^n . Let W_n be any sequence of virtual S_n -representations given by a character polynomial, we compute $H^i(Y_n; W_n)$ for all i and all n in terms of double generating functions. One consequence of the computation is a new recurrence phenomenon: the stable twisted Betti numbers $\lim_{n \rightarrow \infty} \dim H^i(Y_n; W_n)$ are linearly recurrent in i . Our method is to compute twisted point-counts on the \mathbb{F}_q -points of certain algebraic varieties, and then pass through the Grothendieck-Lefschetz fixed point formula to prove results in topology. We also generalize a result of Church-Ellenberg-Farb about the configuration spaces of the affine line to those of a general smooth variety.

1 Introduction

We consider two families of spaces indexed by natural numbers n . The first family is the configuration space of ordered n -tuples of distinct points in a manifold M :

$$\mathrm{PConf}_n M := \{(x_1, \dots, x_n) \in M^n : x_i \neq x_j, \forall i \neq j\}.$$

The symmetric group S_n acts freely on $\mathrm{PConf}_n M$ by permuting the ordered points. The quotient $\mathrm{Conf}_n M := \mathrm{PConf}_n M / S_n$ is the configuration space of *unordered* n -tuples of distinct points. The second family is the space of n linearly independent lines in \mathbb{C}^n :

$$\tilde{\mathcal{T}}_n(\mathbb{C}) := \{(L_1, \dots, L_n) : L_i \text{ a line in } \mathbb{C}^n, L_1, \dots, L_n \text{ linearly independent}\}.$$

S_n acts freely on $\tilde{\mathcal{T}}_n(\mathbb{C})$ by permuting the ordered lines. The quotient $\mathcal{T}_n(\mathbb{C}) := \tilde{\mathcal{T}}_n(\mathbb{C}) / S_n$ can be identified with the space of maximal tori in $\mathrm{GL}_n(\mathbb{C})$. See Section 3 for more details.

Every normal S_n -cover $X \rightarrow Y$ gives a natural bijection between representations of S_n and local systems on Y that become trivial when restricted to X . Thus, every S_n -representations give rise to a local system on $\mathrm{Conf}_n M$ and on $\mathcal{T}_n(\mathbb{C})$.

Question 1 (Twisted Betti numbers). What are the twisted Betti numbers $\dim H^i(\mathrm{Conf}_n M; W_n)$ and $\dim H^i(\mathcal{T}_n(\mathbb{C}); W_n)$ for each i and n , and for each representation W_n of S_n ?

These twisted Betti numbers have geometric, arithmetic, and combinatorial meaning (see *e.g.* Sections 2 and 5 in [F]). The program of computing these numbers dates back to

the work of Arnol'd in the 1960s. For example, if W_n is the trivial, the sign, or the standard representations of S_n , then $\dim H^i(\text{Conf}_n(\mathbb{C}); W_n)$ have been known for all i and n , by the work of Arnol'd [Ar], Cohen [Co], and Vassiliev [Va]. However, even in the special case when $M = \mathbb{C}$, there is no known formula of $\dim H^i(\text{Conf}_n(\mathbb{C}); W_n)$, for every i and n and W_n . In his 2014 ICM talk, Farb proposed a list of problems, one of which (Problem 2.1 in [F]) is equivalent to Question 1. See Remark 1 below for more details.

This paper contains two collections of results: one topological and one arithmetic. We will use the arithmetic results to obtain results in topology.

Topological results:

- Theorem 1 computes $\dim H^i(\text{Conf}_n(\mathbb{C}); W_n)$ and $\dim H^i(\mathcal{T}_n(\mathbb{C}); W_n)$ for all i and all n , and for all representations W_n of S_n . This answers Question 1 for $M = \mathbb{C}$.
- In Corollary 2, we discover a new recurrence phenomenon: the stable twisted Betti numbers $\lim_{n \rightarrow \infty} \dim H^i(\text{Conf}_n(\mathbb{C}); W_n)$ and $\lim_{n \rightarrow \infty} \dim H^i(\mathcal{T}_n(\mathbb{C}); W_n)$ satisfy linear recurrence relations in i .

Arithmetic results:

- Theorem 3 computes weighted point-counts on the \mathbb{F}_q -points of $\text{Conf}_n V$ where V is a smooth variety.
- Corollary 4 states that when $n \rightarrow \infty$, the weighted point-counts on the \mathbb{F}_q -points of $\text{Conf}_n V$ converges in some appropriate sense. This gives a new proof of a recent theorem of Farb-Wolfson and generalizes a theorem of Church-Ellenberg-Farb.

1.1 Computing twisted Betti numbers.

We will consider Question 1 in a more general setting, where W_n is allowed to be a *virtual* S_n -representation, *i.e.* a formal \mathbb{Q} -linear combination of S_n -representations. Virtual representations are in natural bijection with the set of class functions of S_n . In this case, $\dim H^i(\text{Conf}_n(\mathbb{C}); W_n)$ and $\dim H^i(\mathcal{T}_n(\mathbb{C}); W_n)$ are now well-defined rational numbers since the cohomology functor is additive in coefficients.

For each positive integer k , define $X_k : \coprod_{n=1}^{\infty} S_n \rightarrow \mathbb{Z}$ to be the class function with $X_k(\sigma)$ the number of k -cycles in the unique cycle decomposition of $\sigma \in S_n$. A *character polynomial* is a polynomial $P \in \mathbb{Q}[X_1, X_2, \dots]$. It defines a class function on S_n for all n . Define the *degree* of a character polynomial by letting each variable X_k to have degree k . For a sequence of nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_l)$, define a character polynomial by

$$\binom{X}{\lambda} := \binom{X_1}{\lambda_1} \binom{X_2}{\lambda_2} \cdots \binom{X_l}{\lambda_l}.$$

Then $\binom{X}{\lambda}$ has degree $|\lambda| := \sum_{k=1}^l k\lambda_k$. For each fixed n , every class function on S_n is a \mathbb{Q} -linear combination of character polynomials of the form $\binom{X}{\lambda}$. For example, the indicator function on the conjugacy class of $\sigma \in S_n$ is $\binom{X}{\lambda}$ where $\lambda = (X_1(\sigma), \dots, X_n(\sigma))$. Therefore, to answer Question 1, it suffices to consider the case $W_n := \binom{X}{\lambda}$.

Theorem 1 (Generating function for twisted Betti numbers). *Let μ be the classical Möbius function, and let $M_k(z^{-1}) := \frac{1}{k} \sum_{j|k} \mu(\frac{k}{j}) z^{-j}$ be the k -th necklace polynomial in*

z^{-1} . For any sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, we have the following two equations of formal power series in two variables z and t .

$$(I) \quad \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \dim H^i(\text{Conf}_n(\mathbb{C}); \binom{X}{\lambda}) (-z)^i t^n = \frac{1-zt^2}{1-t} \prod_{k=1}^l \binom{M_k(z^{-1})}{\lambda_k} \left(\frac{(tz)^k}{1+(tz)^k} \right)^{\lambda_k}$$

$$(II) \quad \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{\dim H^{2i}(\mathcal{T}_n(\mathbb{C}); \binom{X}{\lambda})}{(1-z)(1-z^2)\cdots(1-z^n)} z^i t^n = \left[\prod_{k=1}^l \frac{1}{\lambda_k!} \left(\frac{t^k}{k(1-z^k)} \right)^{\lambda_k} \right] \cdot \prod_{j=0}^{\infty} \frac{1}{1-tz^j}$$

In (I), all negative power of z in $M_k(z^{-1})$ will cancel with other positive powers of z so that the right-hand-side of the equality is indeed a series in z and t . In (II), we only consider $H^{2i}(\mathcal{T}_n(\mathbb{C}); \binom{X}{\lambda})$ because $H^{2i+1}(\mathcal{T}_n(\mathbb{C}); \binom{X}{\lambda}) = 0$ by the work of Borel [Bo].

Remark 1 (Representation stability). Farb proposed the following problem (Problem 2.1 in [F]): for a manifold M , compute the decomposition of $H^i(\text{PConf}_n M; \mathbb{Q})$ into a sum of irreducible representations of S_n . Remarkably, such a decomposition does not depend on n when n is sufficiently large. This result of *representation stability* was first proved by Church-Farb [CF] for $M = \mathbb{C}$, and later by Church [Chu] for M any connected orientable manifold of finite type (see also [CEF1] for a different proof). Farb proposed a second problem (Problem 3.5 in [F]) of computing the *stable* decomposition of $H^i(\text{PConf}_n M; \mathbb{Q})$ when n is large. Note that for any S_n -representation W_n , the transfer isomorphism associated to the S_n -cover $\text{PConf}_n M \rightarrow \text{Conf}_n M$ gives:

$$\dim H^i(\text{Conf}_n M; W_n) = \langle H^i(\text{PConf}_n M; \mathbb{Q}), W_n \rangle_{S_n}, \quad (1.1)$$

where $\langle U, V \rangle_{S_n}$ stands for the usual inner product of two S_n -representations U and V . Hence, computing the multiplicities of W_n in the decomposition of $H^i(\text{PConf}_n(\mathbb{C}); \mathbb{Q})$ are equivalent to computing twisted Betti numbers of $\text{Conf}_n M$ in W_n .

The simplest nontrivial case for Farb's two questions is when $M = \mathbb{C}$. Theorem 1 (I) reduces Farb's two questions in this case to computing Taylor expansions of rational functions. See Section 2.8 for more discussion and examples.

Remark 2 (Twisted homological stability). Representation stability for $\text{PConf}_n(\mathbb{C})$ implies twisted homological stability for $\text{Conf}_n(\mathbb{C})$. Precisely, Church-Ellenberg-Farb (Theorems 1.9 in [CEF1]) proved that for any character polynomial P and for each fixed i , the twisted Betti numbers $\dim H^i(\text{Conf}_n(\mathbb{C}); P)$ stabilize when n is sufficiently large. Later, Hersh-Reiner gave a different proof of the stability of $\dim H^i(\text{Conf}_n(\mathbb{C}); P)$ with an improved stable range in n (Theorem 4.3 in [HR]). We will give a third proof of this stability result in Corollary 7 using Theorem 1. The implied stable range is a small improvement of that obtained by Hersh-Reiner, and is optimal (see Remark 6 below). The three papers ([CEF1], [HR] and the present one) land at the same result from three totally different points of views respectively: topological, combinatorial, and arithmetic.

Linear recurrence of stable twisted Betti numbers in i . Besides finding new proofs of homological stability, we discover a new phenomenon: the stable cohomology of $\text{Conf}_n(\mathbb{C})$ and $\mathcal{T}_n(\mathbb{C})$ as $n \rightarrow \infty$ with twisted coefficients are linearly recurrent in i .

Corollary 2 (Linear recurrence of stable twisted Betti numbers). Fix an arbitrary character polynomial $P \in \mathbb{Q}[X_1, X_2, \dots]$. Let $N = \deg P$.

(I) For each i , denote $\alpha_i := \lim_{n \rightarrow \infty} \dim H^i(\text{Conf}_n(\mathbb{C}); P)$. There exist integers c_1, \dots, c_N such that for all $i \geq N + 2$,

$$\alpha_i = c_1 \alpha_{i-1} + c_2 \alpha_{i-2} + \dots + c_N \alpha_{i-N}.$$

(II) For each i , denote $\beta_i := \lim_{n \rightarrow \infty} \dim H^{2i}(\mathcal{T}_n(\mathbb{C}); P)$. There exist integers d_1, \dots, d_N such that for all $i \geq N$,

$$\beta_i = d_1 \beta_{i-1} + d_2 \beta_{i-2} + \dots + d_N \beta_{i-N}.$$

For example, if we let $\alpha_i := \lim_{n \rightarrow \infty} \dim H^i(\text{Conf}_n(\mathbb{C}); \bigwedge^2 \mathbb{Q}^{n-1})$ where \mathbb{Q}^{n-1} is the standard representation of S_n , then α_i satisfies the linear recurrence relation:

$$\alpha_i = 2\alpha_{i-1} - 2\alpha_{i-2} + 2\alpha_{i-3} - \alpha_{i-4}.$$

See Section 2.8 for more details.

Remark 3 (Topological proof?). We deduce Corollary 2 from Theorem 1 by explicitly calculating the generating functions of α_i and β_i as rational functions. The proof of Theorem 1 uses point-counting, hence crucially depends on the fact that $\text{Conf}_n(\mathbb{C})$ and $\mathcal{T}_n(\mathbb{C})$ are algebraic varieties. Is there any proof of Corollary 2 using only topology? Are there other examples of recurrent stable twisted Betti numbers in i ?

Method: point-counting over finite fields. The method in this article combines ideas from two beautiful papers: one by Church-Ellenberg-Farb [CEF2] and the other by Fulman [Fu]. Church-Ellenberg-Farb observed that there is a remarkable bridge, provided by the Grothendieck-Lefschetz fixed point theorem in étale cohomology, between cohomology in local coefficients (topology) and weighted point-counts on varieties over finite fields (arithmetic). Furthermore, they apply representation stability in topology to prove that certain weighed point-counts converge. Later, Fulman used a different method to improve the arithmetic calculations stated in [CEF2] and obtained certain “finite n ” formulas. In this paper, we will systematically extend Fulman’s calculations of weighted point-counts, and combine it with the approach of Church-Ellenberg-Farb but in the opposite direction: we use point-counting to compute cohomology.

The idea of using point-counting to study the topology of configuration spaces dates back at least to the work of Lehrer-Kisin [LK], and is also used in Section 4.3 of [CEF2]. Our results are continuations of the theme developed by Lehrer-Kisin and Church-Ellenberg-Farb: structures in the cohomology (*e.g.* stability and recurrence) are often reflected in the arithmetic of corresponding varieties, and *vice versa*.

1.2 Weighted point-counts on configuration spaces of smooth varieties.

Fulman’s method in [Fu] allows us to generalize a result of Church-Ellenberg-Farb as follows. Let $\text{Conf}_n V$ be the configuration space of unordered n -tuples of distinct points on a smooth variety V defined over \mathbb{Z} . When V is the affine line, $\text{Conf}_n \mathbb{A}^1$ is just Conf_n as discussed above¹. In general, every class function of S_n gives a function $\text{Conf}_n V(\mathbb{F}_q) \rightarrow \mathbb{Q}$, which can be viewed as a weighting (see Section 2.1 for more details). The following theorem computes weighted point-counts on $\text{Conf}_n V(\mathbb{F}_q)$ in terms of the zeta function $Z(V, t)$ of V over \mathbb{F}_q .

¹For brevity we will consistently use Conf_n to abbreviate for $\text{Conf}_n \mathbb{A}^1$ throughout the paper.

Theorem 3 (Weighted point-counts on $\text{Conf}_n V$). *Let V be a smooth, connected variety over \mathbb{Z} of positive dimension, and let q be any odd prime power. Let μ be the Möbius function, and define $M_k(V, q) := \frac{1}{k} \sum_{m|k} \mu(\frac{k}{m}) |V(\mathbb{F}_{q^m})|$ for each k . For any sequence of nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_l)$, we have the following equality of formal power series in t :*

$$\sum_{n=0}^{\infty} \left[\sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} \binom{X}{\lambda}(\sigma_C) \right] t^n = \frac{Z(V, t)}{Z(V, t^2)} \cdot \prod_{k=1}^l \binom{M_k(V, q)}{\lambda_k} \left(\frac{t^k}{1+t^k} \right)^{\lambda_k} \quad (1.2)$$

Thanks to Weil conjectures (proved by Dwork, Grothendieck, Deligne *et al.*), $Z(V, t)$ is a rational function in t with a simple pole at $t = q^{-\dim V}$, which is of the smallest absolute value among all other poles or zeros of $Z(V, t)$. By examining the location of poles in the generating sequence (1.2), we see that any point-count on $\text{Conf}_n V(\mathbb{F}_q)$ weighted by a character polynomial converges as $n \rightarrow \infty$ in the following sense.

Corollary 4 (Convergence of weighted point-counts). *With the same assumptions as in Theorem 3 and letting d be the dimension of the variety V , we have:*

(a) *Define $\mathring{Z}(V, t)$ to be the rational function $Z(V, t) \cdot (1 - q^d t)$ in t . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{q^{nd}} \sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} \binom{X}{\lambda}(\sigma_C) = \frac{\mathring{Z}(V, q^{-d})}{Z(V, q^{-2d})} \prod_{k=1}^l \binom{M_k(V, q)}{\lambda_k} \left(\frac{1}{1 + q^{kd}} \right)^{\lambda_k} \quad (1.3)$$

In particular, for any character polynomial P the following limit exists:

$$\lim_{n \rightarrow \infty} \frac{1}{q^{nd}} \sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} P(\sigma_C). \quad (1.4)$$

(b) *The expected value of $\binom{X}{\lambda}$ as a random variable on $\text{Conf}_n V(\mathbb{F}_q)$ converges:*

$$\lim_{n \rightarrow \infty} \frac{1}{|\text{Conf}_n V(\mathbb{F}_q)|} \sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} \binom{X}{\lambda}(\sigma_C) = \prod_{k=1}^l \binom{M_k(V, q)}{\lambda_k} \left(\frac{1}{1 + q^{kd}} \right)^{\lambda_k}$$

Remark 4 (Related works). The convergence of (1.4) in the special case when $V = \mathbb{A}^1$ was first proved by Church-Ellenberg-Farb (Theorem 1 in [CEF2]). Part (a) generalizes their result to a general smooth variety. It concurs with the recent work of Farb-Wolfson, where they extend the topological approach of [CEF2] and gives a different formula for the left-hand-side of (1.3) in terms of the étale cohomology of $\text{PConf}_n V$ (Theorem B in [FW]). Our proof, inspired by the work of Fulman, is different from the topological approach in [CEF2] and [FW]. We obtain not only the asymptotic formula as $n \rightarrow \infty$ (Corollary 4), but also a generating function for all n (Theorem 3).

Remark 5 (Probabilistic interpretation and analogs in number theory). Part (b) of Corollary 4 has the following probabilistic interpretation: the functions X_1, X_2, X_3, \dots , viewed as random variables on $\text{Conf}_n V(\mathbb{F}_q)$, tends to independent random variables with binomial distribution as $n \rightarrow \infty$. This is a geometric analog of the following fact in number theory: the p -adic orders, for p any prime number, of a random integer chosen uniformly from $\{1, 2, \dots, n\}$ tend to be independent random variables with geometric distributions as $n \rightarrow \infty$. More results about weighted point-counts on $\text{Conf}_n V(\mathbb{F}_q)$ (and other related spaces) motivated by this probabilistic point of view will be presented in the forthcoming work of the author [Che].

Acknowledgment. The author thanks Sean Howe, Jenny Wilson, and Rita Jiménez Roland for helpful conversations on the subject. The author is deeply grateful to his advisor Benson Farb, both for his continued support on the project and for his extensive comments on an earlier draft.

2 Cohomology of configuration spaces via point counting

In this section, we will first prove Theorem 3 and Corollary 4 about weighted point-counts on $\text{Conf}_n V(\mathbb{F}_q)$. The main ideas of the proofs were already contained in Fulman's paper [Fu], though he only proved the formulas in the special case when $V = \mathbb{A}^1$ and when $\lambda = (0)$, (1) and $(0, 1)$. We systematically extend Fulman's result to all V and all λ , using some technical input from the Weil conjectures. We then apply the general formula in the case when $V = \mathbb{A}^1$ to prove part (I) of Theorem 1 and of Corollary 2 about $\text{Conf}_n(\mathbb{C})$.

2.1 General set-up

Throughout this section, we will fix V to be a smooth and connected variety over \mathbb{Z} of dimension $d \geq 1$. Define the *configuration space* of V to be the (scheme-theoretic) quotient

$$\text{Conf}_n V := \{(x_1 \cdots, x_n) \in V^n : x_i \neq x_j, \forall i \neq j\} / S_n.$$

where S_n acts on V^n by permuting the coordinates. $\text{Conf}_n V$ is also a variety over \mathbb{Z} (by [Mu], page 66). So we can study its \mathbb{F}_q -points $\text{Conf}_n V(\mathbb{F}_q)$. An element in $\text{Conf}_n V(\mathbb{F}_q)$ is a set of distinct points $C = \{x_1, \dots, x_n\} \subseteq V(\overline{\mathbb{F}}_q)$ such that the Frobenius map $\text{Frob}_q : V(\overline{\mathbb{F}}_q) \rightarrow V(\overline{\mathbb{F}}_q)$ preserves the set. The action of Frob_q on C gives a permutation $\sigma_C \in S_n$, well-defined and unique up to conjugacy. Therefore, any class function χ of S_n gives a well-defined function $\text{Conf}_n V(\mathbb{F}_q) \rightarrow \mathbb{Q}$ by $C \mapsto \chi(\sigma_C)$.

Example: $V = \mathbb{A}^1$. When V is the affine line \mathbb{A}^1 , we use Conf_n to abbreviate for $\text{Conf}_n \mathbb{A}^1$. Elements $C \in \text{Conf}_n(\mathbb{F}_q)$ are in bijection with monic, square-free, degree- n polynomials in $\mathbb{F}_q[x]$ via the map

$$C = \{x_1, \dots, x_n\} \mapsto f_C(x) := (x - x_1) \cdots (x - x_n).$$

Under this bijection, $X_k(\sigma_C)$, defined as the number of k -cycles in σ_C , equals to the number of degree- k factors in the irreducible factorization of $f_C(x)$ over \mathbb{F}_q .

2.2 Proof of Theorem 3

First we recall some basic facts about the zeta function of a variety V over \mathbb{F}_q :

$$Z(V, t) := \prod_x (1 - q^{\deg x})^{-1}$$

where the product is taken over all closed points x on V over \mathbb{F}_q . Weil conjectures give that $Z(V, t)$ is a rational function in t . Let $M_k(V, q)$ denote the number of closed points on V of

degree k , which is equivalently the number of orbits of Frob_q acting on $V(\overline{\mathbb{F}}_q)$ of size k . We have

$$Z(V, t) = \prod_{k=1}^{\infty} (1 - q^k)^{-M_k(V, q)}. \quad (2.1)$$

Note that the fixed points of Frob_q on $V(\overline{\mathbb{F}}_q)$ are precisely $V(\mathbb{F}_q)$. Similarly, for each k we have

$$|V(\mathbb{F}_{q^k})| = \sum_{m|k} m M_m(V, q).$$

By Möbius inversion,

$$M_k(V, t) = \frac{1}{k} \sum_{m|k} \mu\left(\frac{k}{m}\right) |V(\mathbb{F}_{q^m})|.$$

Proof of Theorem 3. Define a formal power series in x_1, \dots, x_l and t :

$$F(x_1, \dots, x_l, t) := \sum_{n=0}^{\infty} \left[\sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} x_1^{X_1(\sigma_C)} x_2^{X_2(\sigma_C)} \dots x_l^{X_l(\sigma_C)} \right] t^n \quad (2.2)$$

Recall that an element $C \in \text{Conf}_n V(\mathbb{F}_q)$ is just a subset of $V(\overline{\mathbb{F}}_q)$ of size n that is preserved by Frob_q . Thus, every $C \in \text{Conf}_n V(\mathbb{F}_q)$ can be decomposed uniquely into a disjoint union of distinct orbits of Frob_q acting on $V(\overline{\mathbb{F}}_q)$. The number of Frob_q -orbits in C of size k is $X_k(\sigma_C)$. The unique decomposition of $C \in \text{Conf}_n V(\mathbb{F}_q)$ into disjoint union of distinct Frob_q -orbits gives the following product formula².

$$\begin{aligned} F(x_1, \dots, x_l, t) &= \left[\prod_{k \geq 1} (1 + t^k)^{M_k(V, q)} \right] \prod_{k \leq l} (1 + x_k t^k)^{M_k(V, q)} \\ &= \left[\prod_{k=1}^{\infty} (1 + t^k)^{M_k(V, q)} \right] \prod_{k \leq l} \left(\frac{1 + x_k t^k}{1 + t^k} \right)^{M_k(V, q)} \\ &= \left[\prod_{k=1}^{\infty} \left(\frac{1 - t^{2k}}{1 - t^k} \right)^{M_k(V, q)} \right] \prod_{k \leq l} \left(\frac{1 + x_k t^k}{1 + t^k} \right)^{M_k(V, q)} \end{aligned}$$

By the product formula (2.1), we obtain

$$F(x_1, \dots, x_l, t) = \frac{Z(V, t)}{Z(V, t^2)} \prod_{k \leq l} \left(\frac{1 + x_k t^k}{1 + t^k} \right)^{M_k(V, q)} \quad (2.3)$$

Next we apply the formal differential operator

$$\left(\frac{\partial}{\partial x} \right)^\lambda := \left(\frac{\partial}{\partial x_1} \right)^{\lambda_1} \left(\frac{\partial}{\partial x_2} \right)^{\lambda_2} \dots \left(\frac{\partial}{\partial x_l} \right)^{\lambda_l}$$

²This is analogous to how unique factorization for integers gives the Euler product formula of Riemann zeta function.

to the series $F(x_1, \dots, x_l, t)$ and then evaluate at $(x_1, \dots, x_l) = (1, \dots, 1)$, obtaining the following equalities. The symbol $\lambda!$ is an abbreviation for $(\lambda_1!)(\lambda_2!) \cdots (\lambda_l!)$. Differentiating (2.2) gives

$$\left(\frac{\partial}{\partial x}\right)^\lambda F(1, \dots, 1, t) = \lambda! \cdot \sum_{n=0}^{\infty} \left(\sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} \binom{X}{\lambda}_{(\sigma_C)} \right) t^n$$

Differentiating (2.3) gives

$$\left(\frac{\partial}{\partial x}\right)^\lambda F(1, \dots, 1, t) = \lambda! \cdot \frac{Z(V, t)}{Z(V, t^2)} \cdot \prod_{k=1}^l \binom{M_k(V, q)}{\lambda_k} \left(\frac{t^k}{1+t^k} \right)^{\lambda_k}$$

Theorem 3 follows by equating these two expressions for $(\frac{\partial}{\partial x})^\lambda F(1, \dots, 1, t)$. \square

2.3 Proof of Corollary 4

First we recall the following basic fact from calculus.

Lemma 5. *Given $A(t) = \sum_{n=0}^{\infty} a_n t^n$ where a_n are real numbers. Suppose $A(t) = H(t)/(1-ct)$ where c is a constant, and the radius of convergence of $H(t)$ is strictly greater than $|c^{-1}|$. Then $\lim_{n \rightarrow \infty} \frac{a_n}{c^n}$ exists and is equal to $H(c^{-1})$.*

Let

$$A(t) := \frac{Z(V, t)}{Z(V, t^2)} \cdot \prod_{k=1}^l \binom{M_k(V, q)}{\lambda_k} \left(\frac{t^k}{1+t^k} \right)^{\lambda_k}$$

The Riemann Hypothesis over finite fields (proved by Deligne [De]) says that $Z(V, t)$ has a simple pole at $t = q^{-d}$ where $d = \dim V$. Moreover, each other zero or pole of $Z(V, t)$ has absolute value q^{-j} for some $j \leq 2d-1$. Thus,

$$\mathring{Z}(V, t) := Z(V, t)(1 - q^d t)$$

has no pole at $|t| < q^{-d+\frac{1}{2}}$; while $1/Z(V, t^2)$ has no pole at $|t| < q^{-2d} < q^{-d}$ (recall that $d = \dim V > 0$). Hence $A(t)$ and $c = q^{-d}$ satisfy the hypothesis of Lemma 5, by which we conclude

$$\lim_{n \rightarrow \infty} \frac{1}{q^{nd}} \sum_{C \in \text{Conf}_n V(\mathbb{F}_q)} \binom{X}{\lambda}_{(\sigma_C)} = \left[A(t)(1 - q^d t) \right]_{t=q^{-d}}$$

This establishes (1.3).

Every character polynomial P is a \mathbb{Q} -linear combination of $\binom{X}{\lambda}$ for different λ . Thus, the limit (1.4) converges for all P . Part (a) is proved.

In the case when $\lambda = (0)$, part (a) gives

$$\lim_{n \rightarrow \infty} \frac{|\text{Conf}_n V(\mathbb{F}_q)|}{q^{nd}} = \frac{\mathring{Z}(V, q^{-d})}{Z(V, q^{-2d})}. \quad (2.4)$$

Part (b) follows by taking the ratio of (1.3) and (2.4). \square

2.4 Connecting arithmetic and topology of Conf_n

For the rest of this paper, we will focus on the case when $V = \mathbb{A}^1$. Recall that we use Conf_n to abbreviate for $\text{Conf}_n \mathbb{A}^1$. Let W be a representation of S_n , with character χ_W . Church-Elzenberg-Farb proved the following equation connecting arithmetic of $\text{Conf}_n(\mathbb{F}_q)$ and topology of $\text{Conf}_n(\mathbb{C})$: (Proposition 4.1 in [CEF2])

$$\sum_{C \in \text{Conf}_n(\mathbb{F}_q)} \chi_W(\sigma_C) = q^n \sum_i (-1)^i \dim H^i(\text{Conf}_n(\mathbb{C}); W) q^{-i}. \quad (2.5)$$

By additivity, same formula holds if we replace W by a virtual representation. See Section 4 in [CEF2] for how (2.5) is obtained from the Grothendieck-Lefschetz fixed point theorem in étale cohomology. Results from the previous section (in the case when $V = \mathbb{A}^1$) give us access to the left-hand-side of (2.5), from which we can prove results about $H^i(\text{Conf}_n(\mathbb{C}); W)$.

2.5 Proof of Theorem 1, (I)

We abbreviate the twisted Betti number as

$$\alpha_i(n) := \dim H^i(\text{Conf}_n(\mathbb{C}); \binom{X}{\lambda}) \quad (2.6)$$

for each i and n . Define the double generating function for $\alpha_i(n)$ as the formal power series in two variables z and t

$$\Phi_\lambda(z, t) := \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \alpha_i(n) (-z)^i t^n \quad (2.7)$$

We want to compute $\Phi_\lambda(z, t)$ as a rational function. We will need the following lemma.

Lemma 6. *Suppose $\Phi(z, t)$ and $\Psi(z, t)$ are two power series in two formal variables z and t . If for every prime power q , we have $\Phi(q^{-1}, t) = \Psi(q^{-1}, t)$ as formal power series in t , then $\Phi(z, t) = \Psi(z, t)$ as formal series in z and t .*

Proof of Lemma. Suppose $\Phi_\lambda(t, z) = \sum_{n=0}^{\infty} \phi_n(z) t^n$ and $\Psi(t, z) = \sum_{n=0}^{\infty} \psi_n(z) t^n$, where $\phi_n(z)$ and $\psi_n(z)$ are formal series in z for each n . By hypothesis, for every prime power q , we have $\phi_n(q^{-1}) = \psi_n(q^{-1})$. Recall the following fact from calculus:

- If an infinite series $h(z) = \sum_{i=0}^{\infty} a_i z^i$ converges at $z = z_0$, then it converges absolutely at all z with $|z| < |z_0|$.

Hence, both $\phi_n(z)$ and $\psi_n(z)$ are holomorphic functions on a disk with a positive radius centered at 0. Since $\phi_n(z) = \psi_n(z)$ for all $z \in \{q^{-1} \mid q \text{ is a prime power}\}$ which accumulates at 0, it must be $\phi_n(z) = \psi_n(z)$ as holomorphic functions. By the uniqueness of power series expansion, $\phi_n(z) = \psi_n(z)$ as formal series in z . Thus $\Phi(z, t) = \Psi(z, t)$ as formal series in z and t . \square

Now we evaluate the double generating function $\Phi_\lambda(z, t)$ at $z = q^{-1}$.

$$\begin{aligned}
\Phi_\lambda(q^{-1}, t) &= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i b_i(n) q^{-i} t^n \\
&= \sum_{n=0}^{\infty} \left[\sum_{C \in \text{Conf}_n(\mathbb{F}_q)} \binom{X}{\lambda}(\sigma_C) \right] (q^{-1}t)^n && \text{By (2.5)} \\
&= \frac{Z(\mathbb{A}^1, tq^{-1})}{Z(\mathbb{A}^1, (tq^{-1})^2)} \prod_{k=1}^n \binom{M_k(\mathbb{A}^1, q)}{\lambda_k} \left(\frac{(tq^{-1})^k}{1 + (tq^{-1})^k} \right)^{\lambda_k} && \text{By Theorem 3}
\end{aligned}$$

The k -th necklace polynomial in x is

$$M_k(x) = \frac{1}{k} \sum_{m|k} \mu\left(\frac{k}{m}\right) x^m.$$

A standard calculation gives that $Z(\mathbb{A}^1, t) = \frac{1}{1-qt}$, and that $M_k(\mathbb{A}^1, q) = M_k(q)$. Thus, we simplify the above:

$$\Phi_\lambda(q^{-1}, t) = \frac{1 - t^2 q^{-1}}{1 - t} \prod_{k=1}^n \binom{M_k(q)}{\lambda_k} \left(\frac{(tq^{-1})^k}{1 + (tq^{-1})^k} \right)^{\lambda_k} \quad (2.8)$$

Since (2.8) holds at $z = q^{-1}$ for any prime power q . By Lemma 6, the same equation holds when q^{-1} is replaced by a formal variable z . □

2.6 Stability of Betti numbers

It was known by the general theory of representation stability developed by Church-Ellenberg-Farb that for any character polynomial P , the twisted Betti numbers $\dim H^i(\text{Conf}_n(\mathbb{C}); P)$ will be independent of n when n is sufficiently large. We will give a different proof of this result with an improved stability range for n .

Corollary 7. *For every character polynomial P and for every i , we have*

$$\dim H^i(\text{Conf}_n(\mathbb{C}); P) = \dim H^i(\text{Conf}_{n+1}(\mathbb{C}); P) \quad (2.9)$$

when $n \geq i + \deg P + 1$.

Remark 6. Church-Ellenberg-Farb first proved (2.9) when $n \geq 2i + \deg P$ (Theorem 1 [CEF2]). Later, Hersh-Reiner gave a different proof of (2.9) with a better stable range: $n \geq \max\{2 \deg P, \deg P + i + 1\}$ (Theorem 4.3 in [HR]). The stable range in Corollary 7 is a small improvement of the range obtained by Hersh-Reiner, and is sharp, as we will show it in Section 2.8.

Proof. It suffices to consider when $P = \binom{X}{\lambda}$ for some sequence $\lambda = (\lambda_1, \dots, \lambda_k)$. In this case $\deg \binom{X}{\lambda} = \sum_k k \lambda_k$. Let $\alpha_i(n)$ be as in (2.6), and let $\Phi_\lambda(z, t)$ be as in (2.7).

$$(1 - t)\Phi_\lambda(t, z) = 1 + t \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} [\alpha_i(n+1) - \alpha_i(n)] z^i t^n$$

It suffices to check that $(1-t)\Phi_\lambda(t, z)$ is a sum of monomials of the form $z^i t^n$ where $n-i \leq \sum_{k=0}^l k\lambda_k + 1$.

We will say an infinite series in z and t has slope $\leq m$ if it is a sum of monomials $z^i t^n$ where $n-i \leq m$. We want to show that the series given by Theorem 1

$$(1-t)\Phi_\lambda(t, z) = (1-zt^2) \prod_{k=1}^l \binom{M_k(z^{-1})}{\lambda_k} \left(\frac{(tz)^k}{1+(z)^k} \right)^{\lambda_k} \quad (2.10)$$

has slope $\leq \sum_{k=0}^l k\lambda_k + 1$. We analyze each factor.

- $(1-zt^2)$ has slope $\leq 2-1=1$.
- For each k , the factor $M_k(z^{-1})$ has slope $\leq k$. Thus, $\binom{M_k(z^{-1})}{\lambda_k}$ has slope $\leq k\lambda_k$.
- For each k , $\left[\frac{(tz)^k}{1+(tz)^k} \right]^{\lambda_k}$ has slope ≤ 0 .

Therefore, the product in (2.10) has slope $\leq 1 + \sum_{k=0}^l k\lambda_k$. This establishes the corollary. \square

2.7 Proof of Corollary 2, (I).

Let α_i be $\alpha_i(n)$ when $n \geq i + |\lambda| + 1$ in the stable range. Define the generating function

$$\Phi_\lambda^\infty(z) := \sum_{i=0}^{\infty} \alpha_i(-z)^i$$

By Lemma 5, we can calculate $\Phi_\lambda^\infty(z)$ using $\Phi_\lambda(z, t)$:

$$\begin{aligned} \Phi_\lambda^\infty(z) &= \left[(1-t)\Phi_\lambda(z, t) \right]_{t=1} && \text{by Theorem 1} \\ &= (1-z) \prod_{k=1}^l \binom{M_k(z^{-1})}{\lambda_k} \left(\frac{z^k}{1+z^k} \right)^{\lambda_k} \end{aligned} \quad (2.11)$$

In particular, $\Phi_\lambda^\infty(z)$ in (2.11) is a rational function in z . The denominator is a polynomial in z of degree $\sum_{k=1}^l k\lambda_k = |\lambda|$. The numerator has degree at most $1 + |\lambda|$. This implies that α_i satisfies a linear recurrence relation of length $|\lambda|$ once $i > |\lambda| + 1$. \square

2.8 Examples

Recall that irreducible representations of S_n are in bijection with partitions of n . For a fixed partition $\mu \vdash n$ where $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$, we will denote by $V(\mu)_n$ ³ the representation of S_n corresponding to the partition $n = (n - \sum_{i=1}^r \mu_i) + \mu_1 + \cdots + \mu_r$ for all n sufficiently large, i.e. for $n - \sum_{i=1}^r \mu_i \geq \mu_1$. Going from $V(\mu)_n$ to $V(\mu)_{n+1}$ corresponds to

³Sometimes we will suppress n from the notation.

adding one block in the first row of the corresponding Young diagram. Church-Ellenberg-Farb proved that $H^i(\text{PConf}_n(\mathbb{C}); \mathbb{Q})$ is *multiplicity stable* (Theorem 1.9 in [CEF1]): for each i , there is a finite set Q_i of partitions such that

$$H^i(\text{PConf}_n(\mathbb{C}); \mathbb{Q}) \cong \bigoplus_{\mu \in Q_i} V(\mu)_n^{\oplus d_i(\mu)}$$

for all n sufficiently large. In particular, the sum over Q_i is independent of n . Farb proposed the problem of computing $d_i(\mu)$ for each i and each μ (Problem 3.5 in [F]). Macdonald proved that for all partition μ , the character of $V(\mu)_n$ is given by a unique character polynomial P_μ for all n sufficiently large (Example I.7.14 in [Ma]). Therefore, by transfer (1.1), computing $d_i(\mu)$ is equivalent to computing the stable cohomology of $H^i(\text{Conf}_n(\mathbb{C}); P_\mu)$. We will demonstrate the case of computing these using Theorem 1 (I) in three examples where μ is the partition $1 = 1$, or $2 = 1 + 1$, or $2 = 2$.

Example 1: $W_n = V(1)_n$. Assume $n \geq 2$, the irreducible representation $V(1)_n$ corresponds to the Young diagram $(n - 1, 1)$. It is also known as the *standard representation*:

$$V(1)_n \cong \{(x_1, \dots, x_n) \mid \sum x_i = 0\} \cong \mathbb{Q}^{n-1}$$

where S_n acts by permuting the coordinates.

The S_n -character of W is given by the character polynomial $X_1 - 1$. If we abbreviate the Betti number as

$$\alpha_i(n) = \dim H^i(\text{Conf}_n(\mathbb{C}); V(1)_n),$$

then Theorem 1 gives that the double generating function of $\alpha_i(n)$ is

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \alpha_i(n) z^i t^n &= \frac{1-t^2 z}{1-t} \left[\frac{t}{1+tz} - 1 \right] \\ &= (-z + z^2)t^3 + (-z + 2z^2 - z^3)t^4 + (-z + 2z^2 - 2z^3 + z^4)t^5 \\ &\quad + (-z + 2z^2 - 2z^3 + 2z^4 + z^5)t^6 + \dots \end{aligned}$$

Thus, we conclude that when $n \geq 3$,

$$\alpha_i(n) = \begin{cases} 0 & i = 0 \\ 1 & i = 1 \\ 2 & 0 < i < n - 1 \\ 1 & i = n - 1 \end{cases}$$

Remark 7. A computation of $\dim H^i(\text{Conf}_n(\mathbb{C}); V(1)_n)$ from Lehrer-Solomon's description of $H^i(\text{PConf}_n(\mathbb{C}); \mathbb{Q})$ was presented in Proposition 4.5 of [CEF2]. It took about one and half pages. The computation above using generating function is a faster procedure.

The stable Betti numbers are:

$$\alpha_i := \lim_{n \rightarrow \infty} \dim H^i(\text{Conf}_n(\mathbb{C}); V(1)) = \begin{cases} 0 & i = 0 \\ 1 & i = 1 \\ 2 & i > 1 \end{cases}$$

When $i \geq 2$, the stable Betti numbers α_i are the same, which in particular satisfy a recurrence relation of length 1. From this example we see that the bounds in Corollary 2 (I) and Corollary 7 are sharp.

Example 2: $W_n = V(\mathbf{1}, \mathbf{1})_n$. Assume $n \geq 3$, the irreducible representation $V(1, 1)_n$ corresponds to the Young diagram $(n - 2, 1, 1)$. The dimension of $V(1, 1)$ is $(n^2 - 3n + 2)/2$. In fact, we have $V(1, 1) \cong \bigwedge^2 \mathbb{Q}^{n-1}$ where \mathbb{Q}^{n-1} is the standard representation $V(1)$. The character of $V(1, 1)$ is given by the following character polynomial:

$$\binom{X_1}{2} - X_1 - X_2 + 1$$

If we abbreviate the Betti numbers $\alpha_i(n) = \dim H^i(\text{Conf}_n(\mathbb{C}); V(1, 1)_n)$, then Theorem 1 gives that

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \alpha_i(n) z^i t^n &= \Phi_{(2)}(z, t) - \Phi_{(1)}(z, t) - \Phi_{(0,1)}(z, t) + \Phi_{(0)}(z, t) \\ &= \frac{1-t^2z}{1-t} \left[\frac{(1-z)t^2}{2(1+tz)^2} - \frac{t}{1+tz} - \frac{(1-z)t^2}{2(1+(tz)^2)} + 1 \right] \end{aligned}$$

By expanding the generating function, we have the following table of the Betti numbers:

$\alpha_i(n) = \dim H^i(\text{Conf}_n(\mathbb{C}); V(1, 1)_n)$												
(i, n)	$n = 3$	4	5	6	7	8	9	10	11	12	13	14
$i = 0$	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	2	2	2	2	2	2	2	2	2	2
3		1	3	5	5	5	5	5	5	5	5	5
4			1	4	6	6	6	6	6	6	6	6
5				1	5	7	7	7	7	7	7	7
6					2	7	10	10	10	10	10	10
7						3	9	13	13	13	13	13
8							3	10	14	14	14	14
9								3	11	15	15	15
10									4	13	18	18
11										5	15	21
12											5	16
13												5

The bold entries lie on the line $n = i + 3$. In each row, the Betti number stabilizes as $n \geq i + 3$. This agrees with the stability bound as predicted in Corollary 2.9: $n > i + \deg(\binom{X_1}{2} - X_1 - X_2 + 1) = i + 2$. Moreover, we can see from the table that the bound is sharp.

Furthermore, from (2.11), we have the following formula for the generating function of the stable Betti numbers $\alpha_i := \lim_{n \rightarrow \infty} \dim H^i(\text{Conf}_n(\mathbb{C}); V(1, 1)_n)$:

$$\begin{aligned} \sum_{i=0}^{\infty} (-1)^i \alpha_i z^i &= \Phi_{(2)}^{\infty}(z) - \Phi_{(1)}^{\infty}(z) - \Phi_{(0,1)}^{\infty}(z) + \Phi_{(0)}^{\infty}(z) = (1-z) \left[\frac{1-z}{2(1+z)^2} - \frac{1}{1+z} - \frac{1-z}{2(1+z^2)} + 1 \right] \\ &= 2z^2 - 5z^3 + 6z^4 - 7z^5 + 10z^6 - 13z^7 + 14z^8 - 15z^9 + 18z^{10} - 21z^{11} + \dots \end{aligned}$$

The stable Betti numbers satisfy the linear recurrence relation:

$$\alpha_i = 2\alpha_{i-1} - 2\alpha_{i-2} + 2\alpha_{i-3} - \alpha_{i-4}.$$

By explicitly solving the recurrence relation, we have $\alpha_0 = 0$, $\alpha_1 = 2$, and when $i \geq 3$,

$$\alpha_i = \begin{cases} 2i - 2 & i \equiv 0 \pmod{4} \\ 2i - 3 & i \equiv 1 \pmod{4} \\ 2i - 2 & i \equiv 2 \pmod{4} \\ 2i - 1 & i \equiv 3 \pmod{4} \end{cases}$$

Remark 8. In Section 4.4 of [CEF2], Church-Ellenberg-Farb used L -functions to compute the stable cohomology of $H^i(\text{Conf}_n(\mathbb{C}); \bigwedge^2 \mathbb{Q}^n)$. Since $\bigwedge^2 \mathbb{Q}^n \cong \bigwedge^2 \mathbb{Q}^{n-1} \oplus \mathbb{Q}^n$, we recover their computation. Moreover, we also obtained unstable cohomology.

Example 3: $W_n = V(2)_n$. Assume $n \geq 4$, the irreducible representation $V(2)_n$ corresponds to the Young diagram $(n-2, 2)$. The dimension of $V(2)$ is $(n^2 - 3n)/2$. In fact, $V(2)$ is a direct summand in the symmetric square of the standard representation \mathbb{Q}^{n-1} . More precisely, we have

$$\text{Sym}^2(\mathbb{Q}^{n-1}) \cong \mathbb{Q}^n \oplus V(2)$$

The character of $V(2)$ is given by the following character polynomial

$$\binom{X_1}{2} + X_2 - X_1$$

If we abbreviate the Betti numbers $\alpha_i(n) := \dim H^i(\text{Conf}_n(\mathbb{C}); V(2)_n)$, Theorem 1 gives

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \alpha_i(n) z^i t^n &= \Phi_{(2)}(z, t) + \Phi_{(0,1)}(z, t) - \Phi_{(1)}(z, t) \\ &= \frac{1-t^2 z}{1-t} \left[\frac{(1-z)t^2}{2(1+tz)^2} + \frac{(1-z)t^2}{2(1+(tz)^2)} - \frac{t}{1+tz} \right] \end{aligned}$$

By expanding the generating function, we have the following table of Betti numbers:

$\alpha_i(n) = \dim H^i(\text{Conf}_n(\mathbb{C}); V(2)_n)$											
(i, n)	$n = 4$	5	6	7	8	9	10	11	12	13	14
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2
3	0	2	3	3	3	3	3	3	3	3	3
4		1	4	6	6	6	6	6	6	6	6
5			2	6	9	9	9	9	9	9	9
6				2	7	10	10	10	10	10	10
7					2	8	11	11	11	11	11
8						3	10	14	14	14	14
9							4	12	17	17	17
10								4	13	18	18
11									4	14	19
12										5	16
13											6

The bold entries lie on the line $n = i + 3$. In each row, the Betti number stabilizes when $n \geq i + 3$. This agrees with the stability bound as predicted in Corollary 2.9: $n > i + \deg(\binom{X_1}{2} + X_2 - X_1) = i + 2$. We can see from the table that the bound is sharp.

Furthermore, from (2.11), we have the following formula for the generating function of the stable Betti numbers:

$$\begin{aligned} \sum_{i=0}^{\infty} (-1)^i \alpha_i z^i &= \Phi_{(2)}^{\infty}(z) - \Phi_{(1)}^{\infty}(z) - \Phi_{(0,1)}^{\infty}(z) + \Phi_{(0)}^{\infty}(z) = (1-z) \left[\frac{1-z}{2(1+z)^2} + \frac{1-z}{2(1+z^2)} - \frac{1}{1+z} \right] \\ &= -z + 2z^2 - 3z^3 + 6z^4 - 9z^5 + 10z^6 - 11z^7 + 14z^8 - 17z^9 + 18z^{10} - 19z^{11} + \dots \end{aligned}$$

The stable Betti numbers satisfies the linear recurrence relation:

$$\alpha_i = 2\alpha_{i-1} - 2\alpha_{i-2} + 2\alpha_{i-3} - \alpha_{i-4}.$$

We can explicitly solve the recurrence relation and obtain that $\alpha_0 = 0$, and when $i \geq 1$,

$$\alpha_i := \lim_{n \rightarrow \infty} \dim H^i(\text{Conf}_n(\mathbb{C}); V(2)) = \begin{cases} 2i-2 & i \equiv 0 \pmod{4} \\ 2i-1 & i \equiv 1 \pmod{4} \\ 2i-2 & i \equiv 2 \pmod{4} \\ 2i-3 & i \equiv 3 \pmod{4} \end{cases}$$

3 Cohomology of $\mathcal{T}_n(\mathbb{C})$ via point counting

In this section we prove part (II) of Theorem 1 and Corollary 2. Our analysis of \mathcal{T}_n closely parallels that of Conf_n before.

3.1 General set-up

$\mathcal{T}_n = \tilde{\mathcal{T}}_n/S_n$ is a scheme over \mathbb{Z} (again, see page 66 in [Mu]). The \mathbb{F}_q -points $\mathcal{T}_n(\mathbb{F}_q)$ consists of sets $L = \{L_1, \dots, L_n\}$ of n linearly independent lines in $\mathbb{P}^{n-1}(\overline{\mathbb{F}}_q)$ such that the Frobenius map $\text{Frob}_q : \mathbb{P}^{n-1}(\overline{\mathbb{F}}_q) \rightarrow \mathbb{P}^{n-1}(\overline{\mathbb{F}}_q)$ preserves the set L .

Let F abbreviate the Frobenius map. An F -stable *torus* in $\text{GL}_n(\mathbb{F}_q)$ is an algebraic subgroup which becomes diagonalizable over $\overline{\mathbb{F}}_q$. An F -stable torus is *maximal* if it is not properly contained in any larger one. Given any F -stable maximal torus T , its n eigenvectors in $\overline{\mathbb{F}}_q^n$ defines a set L_T of n independent lines in $\overline{\mathbb{F}}_q^n$. Thus L_T is a element of $\mathcal{T}_n(\mathbb{F}_q)$. The map $T \mapsto L_T$ gives a bijection between F -stable maximal tori in $\text{GL}_n(\mathbb{F}_q)$ and \mathbb{F}_q -points of \mathcal{T}_n . Therefore, $\mathcal{T}_n(\mathbb{F}_q)$ is precisely the set of F -stable maximal tori in $\text{GL}_n(\mathbb{F}_q)$. See Section 5.1 of [CEF2] for a proof.

For any $T \in \mathcal{T}_n(\mathbb{F}_q)$, the action of Frob_q on L_T , a set of n lines in $\overline{\mathbb{F}}_q^n$, gives a permutation $\sigma_T \in S_n$, unique up to conjugacy. Church-Elleberg-Farb proved the following equation using the Grothendieck-Lefschetz fixed point formula. Given any S_n -representation W with character χ_W ,

$$\sum_{T \in \mathcal{T}_n(\mathbb{F}_q)} \chi_W(\sigma_T) = q^{n(n-1)} \sum_{i=0}^{n(n-1)/2} \dim H^{2i}(\mathcal{T}_n(\mathbb{C}); W) q^{-i} \quad (3.1)$$

This formula was stated in Theorem 5.3 in [CEF2]. By additivity, the same formula holds when W is taken to be a virtual representation of S_n .

3.2 Arithmetic statistics for F -stable maximal tori in $\mathrm{GL}_n(\mathbb{F}_q)$

In this subsection, we will compute the left-hand-side of (3.1) when W is given by a character polynomial of the form $\binom{X}{\lambda}$. Our approach will be a systematic extension of Fulman's method in [Fu]. All the ideas in this subsection were already in Fulman's paper.

Proposition 8. *For each fixed sequence of nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_l)$, let $z_\lambda := \prod_{k=1}^l \lambda_k! k^{\lambda_k}$. We have the following equation of formal power series in t .*

$$\sum_{n=0}^{\infty} \left[\sum_{T \in \mathcal{T}_n(\mathbb{F}_q)} \binom{X}{\lambda}(\sigma_T) \right] \frac{t^n}{|\mathrm{GL}_n(\mathbb{F}_q)|} = \frac{1}{z_\lambda} \left[\prod_{k=1}^l \left(\frac{q^{-k} t^k}{1 - q^{-k}} \right)^{\lambda_k} \right] \cdot \left[\prod_{i=1}^{\infty} \frac{1}{1 - q^{-i} t} \right] \quad (3.2)$$

Proof. We will use the following result of Fulman (stated as Theorem 3.2 in [Fu]).

Theorem (Fulman). *With the notation as above,*

$$\sum_{n=0}^{\infty} \left[\sum_{T \in \mathcal{T}_n(\mathbb{F}_q)} \prod_{i=1}^n x_i^{X_i(\sigma_T)} \right] \frac{t^n}{|\mathrm{GL}_n(\mathbb{F}_q)|} = \prod_{k=1}^{\infty} \exp \left[\frac{x_k t^k}{(q^k - 1)k} \right] \quad (3.3)$$

Let $F(\vec{x}, t)$ denote both sides of (3.3) as a formal power series in infinitely many variables t and x_1, x_2, \dots . We apply the formal differential operator

$$\left(\frac{\partial}{\partial x} \right)^\lambda := \left(\frac{\partial}{\partial x_1} \right)^{\lambda_1} \left(\frac{\partial}{\partial x_2} \right)^{\lambda_2} \dots \left(\frac{\partial}{\partial x_l} \right)^{\lambda_l}$$

to the series $F(\vec{x}, t)$ and then evaluate at $x_i = 1$ for all i . Let $\lambda!$ be an abbreviation for $(\lambda_1!)(\lambda_2!) \dots (\lambda_l!)$. Then

$$\begin{aligned} \lambda! \sum_{n=0}^{\infty} \left[\sum_{T \in \mathcal{T}_n(\mathbb{F}_q)} \binom{X}{\lambda}(\sigma_T) \right] \frac{t^n}{|\mathrm{GL}_n(\mathbb{F}_q)|} &= \left(\frac{\partial}{\partial x} \right)^\lambda \left[F(\vec{x}, t) \right]_{x_i=1, \forall i} \\ &= \left(\frac{\partial}{\partial x} \right)^\lambda \left[\prod_{k=1}^{\infty} \exp \frac{x_k t^k}{(q^k - 1)k} \right]_{x_i=1, \forall i} \\ &= \left[\prod_{k=1}^l \left(\frac{t^k}{(q^k - 1)k} \right)^{\lambda_k} \right] \cdot \left[\prod_{k=1}^{\infty} \exp \frac{t^k}{(q^k - 1)k} \right] \\ &= \left[\prod_{k=1}^l \left(\frac{t^k}{(q^k - 1)k} \right)^{\lambda_k} \right] \cdot \left[\prod_{i=1}^{\infty} \frac{1}{1 - q^{-i} t} \right] \end{aligned}$$

where the last equality follows from

$$\prod_{k=1}^{\infty} \exp \frac{t^k}{(q^k - 1)k} = \prod_{i=1}^{\infty} \frac{1}{1 - q^{-i} t}$$

which can be proved by expanding both sides into power series. \square

3.3 Proof of Theorem 1, (II)

For each i and n , we abbreviate the twisted Betti number as

$$\beta_i(n) := \dim H^{2i}(\mathcal{T}_n(\mathbb{C}); \binom{X}{\lambda}) \quad (3.4)$$

Define a formal power series in z and t

$$\Psi_\lambda(z, t) := \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{\beta_i(n)}{(1-z)(1-z^2)\cdots(1-z^n)} z^i t^n \quad (3.5)$$

We evaluate $\Psi_\lambda(z, t)$ at $z = q^{-1}$:

$$\begin{aligned} \Psi_\lambda(q^{-1}, t) &= \sum_{n=0}^{\infty} \frac{1}{(1-q^{-1})(1-q^{-2})\cdots(1-q^{-n})} \sum_{i=0}^{\infty} \beta_i(n) q^{-i} t^n \\ &= \sum_{n=0}^{\infty} \frac{1}{(q^n - q^{n-1})(q^n - q^{n-2})\cdots(q^n - 1)} \left[q^{n(n-1)} \sum_{i=0}^{\infty} \beta_i(n) q^{-i} (tq)^n \right] \\ &= \sum_{n=0}^{\infty} \frac{1}{|\mathrm{GL}_n(\mathbb{F}_q)|} \sum_{n=0}^{\infty} \left[\sum_{T \in \mathcal{T}_n(\mathbb{F}_q)} \binom{X}{\lambda}(\sigma_T) \right] (tq)^n \quad \text{by 3.1} \\ &= \frac{1}{z_\lambda} \left[\prod_{k=1}^l \left(\frac{t^k}{1-q^{-k}} \right)^{\lambda_k} \right] \cdot \left[\prod_{i=1}^{\infty} \frac{1}{1-q^{1-k}t} \right] \quad \text{by Proposition 8} \end{aligned}$$

Since the equality holds for all prime powers q , the equality also holds when q^{-1} is replaced by a formal variable z by Lemma 6. \square

3.4 Proof of Corollary 2, (II).

As before, it suffices to consider when $P = \binom{X}{\lambda}$. Let $\beta_i(n)$ be as in (3.4) and let β_i be $\lim_{n \rightarrow \infty} \beta_i(n)$, then we have

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{n(n-1)} \frac{\beta_i(n)}{(1-z)(1-z^2)\cdots(1-z^n)} z^i = \sum_{i=0}^{\infty} \frac{\beta_i}{\prod_{j=1}^{\infty} (1-z^j)} z^i \quad (3.6)$$

On the other hand, by Lemma 5, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=0}^{n(n-1)} \frac{\beta_i(n)}{(1-z)(1-z^2)\cdots(1-z^n)} z^i &= \left[(1-t)\Psi_\lambda(z, t) \right]_{t=1} \\ &= \frac{1}{z_\lambda} \prod_{k=1}^l \left(\frac{1}{1-z^k} \right)^{\lambda_k} \cdot \prod_{j=1}^{\infty} \frac{1}{1-z^j} \quad (3.7) \end{aligned}$$

Equating (3.6) and (3.7), we have

$$\sum_{i=0}^{\infty} \beta_i z^i = \frac{1}{z_\lambda} \prod_{k=1}^l \left(\frac{1}{1-z^k} \right)^{\lambda_k}.$$

The generating function for β_i is a rational function in z with denominator a polynomial of degree $|\lambda|$. Thus β_i satisfies a linear recurrence relation of length $|\lambda|$. □

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